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TWO THEOREMS ON STATICAL INDETERMINACY OF STRUCTURES

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Abstract—Classical structural theorems are concerned with extremum conditions of energy as a function of displacement degrees of freedom or that of the complementary energy as a function of member forces. In this brief note the extremum conditions of energy and complementary energy are viewed from the perspective of degrees of indeterminacy in the system. Specifically, it is shown that the complementary energy in a system, under prescribed displacements, is reduced as the statical indeterminacy of the systems is lowered by removal of some redundant members. Further, it is shown that the loss in the complementary energy of the system is greater than the complementary energy in the removed redundant members.

INTRODUCTION

The force method of structural analysis provides an effective means for the introduction of the concept of indeterminacy in structures and it highlights the distinct roles of equilibrium, constitutive relationships and compatibility in the overall solution process. Furthermore, the notion of converting a statically indeterminate structure to a determinate one by releasing the redundant members provides good physical insight into the analysis and in a natural manner distinguishes mechanisms from structures. Normally Castigliano's second theorem is invoked to compute the deflections associated with the set of equilibrating forces, and by equating the deflections associated with the redundant members to given values, the imagined gaps in the structure are closed. This approach is described and illustrated in many introductory texts in structural mechanics [see, e.g., Tong and Christiano (1987), Laible (1985) and Tauchert (1974)]. An equivalent alternative approach and one that we will use, is based on the complementary virtual work principle.

In this note we introduce two theorems related to statical indeterminacy of structures. Although these theorems apply to general elastostatic systems, they are best illustrated through applications to structural systems with a finite number of degrees of statical indeterminacy.

THEOREM 1

This states that the complementary strain energy of a statically indeterminate system subjected to some prescribed displacements, will be reduced (or at best remain unchanged) if the degree of statical indeterminacy of the system is reduced.

In the following we illustrate this theorem and subsequently provide a proof. By choosing a particularly simple example we obtain analytical solutions and thereby maintain the essence of the theorem in the foreground.

Consider the static system shown in Fig. 1. The system is subjected to the prescribed displacement d_1. The three elastic members are linear with stiffnesses k_1, k_2 and k_3 and members 1 and 2 have the same length. The equilibrium equations are:

At joint 1:

$$F_3 = F_1 + F_2. \tag{1}$$

At joint 2:

$$F_1 + F_2 = R_1, \tag{2}$$

where R_1 is the reaction force associated with d_1 and is an unknown of the problem.

There are two equilibrium equations and four unknown forces—hence the system has two degrees of statical indeterminacy with respect to the prescribed displacement.

Now from the complementary virtual work theorem we have

$$\delta\{F\}^{\mathrm{T}}\Delta\{e\} - \delta\{R\}^{\mathrm{T}}\{d\} = 0 \tag{3}$$

where $\{F\}$ is the vector of member forces, $\Delta\{e\}$ is the vector of changes in the extensions of the elastic members, $\{d\}$ denotes the prescribed displacement vector and $\{R\}$ is the vector of reactions associated with the prescribed displacements. For the complementary virtual work expression in eqn (3) the internal forces $\{F\}$ and reactions $\{R\}$ must satisfy the system equilibrium equations.

The elastic constitutive relationship allows us to express the complementary virtual work of the extensions as variations of the complementary strain energy. Thus with the relationship

$$\Delta\{e\} = [\phi]\{F\} \tag{4}$$

where $[\phi]$ is the compliance matrix, we can write

$$\delta\{F\}^{\mathrm{T}}\Delta\{e\} = \delta\{F\}^{\mathrm{T}}[\phi]\{F\} = \delta V^*$$

or

$$V^* = \tfrac{1}{2}\{F\}^{\mathrm{T}}[\phi]\{F\}. \tag{5}$$

In this case we may express eqn (1) as

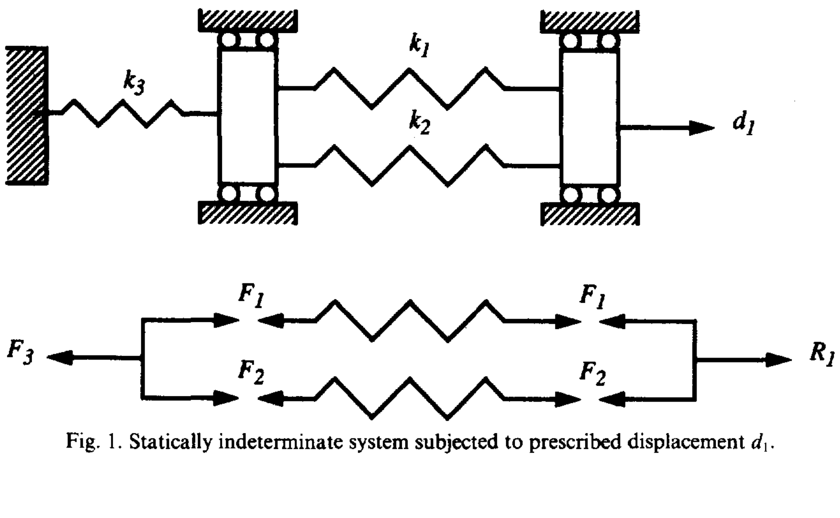

Fig. 1. Statically indeterminate system subjected to prescribed displacement d_1.

$$\delta V^* - \delta W_c = 0, \tag{6}$$

where δW_c denotes the complementary virtual work of prescribed displacements. In the variational statement in eqn (6), equilibrating forces constitute the competing functions and the extremum conditions emerge as a set of compatibility equations.

Now using the equilibrium equations (1) and (2), we express the complementary strain energy of the system in Fig. 1 in terms of F_1 and F_2 as

$$V^* = \tfrac{1}{2}[F_1 \;\; F_2] \begin{bmatrix} \dfrac{1}{k_1}+\dfrac{1}{k_3} & \dfrac{1}{k_3} \\ \dfrac{1}{k_3} & \dfrac{1}{k_2}+\dfrac{1}{k_3} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \tag{7}$$

where the square matrix in eqn (7) is the compliance matrix. Again, in terms of equilibrating forces we may express the complementary virtual work of the prescribed displacement as:

$$\delta W_c = \delta[F_1 \;\; F_2] \begin{bmatrix} d_1 \\ d_1 \end{bmatrix}. \tag{8}$$

Using the above expressions for V* and δW$_c$ in equation (6) we may obtain the following compatibility equations as the system equations

$$\begin{bmatrix} \dfrac{1}{k_1}+\dfrac{1}{k_3} & \dfrac{1}{k_3} \\ \dfrac{1}{k_3} & \dfrac{1}{k_2}+\dfrac{1}{k_3} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_1 \end{bmatrix}. \tag{9}$$

Solving for F_1 and F_2 we obtain:

$$F_1 = \frac{k_3 k_1 d_1}{(k_1+k_2+k_3)}, \tag{10}$$

$$F_2 = \frac{k_3 k_2 d_1}{(k_1+k_2+k_3)}. \tag{11}$$

The member force F_3 and the reaction force R_1 can now be determined from the equilibrium equations and are given by

$$F_3 = R_1 = \frac{k_3(k_1+k_2)d_1}{k_1+k_2+k_3}. \tag{12}$$

The member extensions can also be computed as

$$\Delta e_1 = \frac{F_1}{k_1} = \frac{k_3 d_1}{k_1+k_2+k_3}, \tag{13}$$

$$\Delta e_2 = \frac{F_2}{k_2} = \frac{k_3 d_1}{k_1+k_2+k_3}, \tag{14}$$

$$\Delta e_3 = \frac{F_3}{k_3} = \frac{F_1+F_2}{k_3} = \frac{(k_1+k_2)d_1}{k_1+k_2+k_3}. \tag{15}$$

Finally, the complementary strain energy of the system may be determined via eqns (7), (10) and (11) and is given by

$$V^* = \frac{1}{2}\frac{k_3(k_1+k_2)d_1^2}{k_1+k_2+k_3}. \tag{16}$$

Now theorem 1 states that if the statical indeterminacy of the system is reduced, yielding a system with a complementary strain energy V^*_{new}, then

$$V^*_{\text{new}} \leqslant V^*. \tag{17}$$

To illustrate this, let us remove the first member and reduce the degree of indeterminacy of the system to 1. This can be accomplished readily by setting $k_1 = 0$. Then

$$V^*_{\text{new}} = \frac{1}{2}\frac{k_3 k_2 d_1^2}{k_2+k_3}. \tag{18}$$

To demonstrate the validity of eqn (17), we need to show that

$$\frac{k_1+k_2}{k_1+k_2+k_3} > \frac{k_2}{k_2+k_3}. \tag{19}$$

On rearranging we can write this expression as

$$(k_2+k_3)(k_1+k_2) > k_2(k_1+k_2+k_3),$$

i.e.

$$k_1 \cdot k_3 > 0, \tag{20}$$

which evidently is true since k_1 and k_3 are both positive.

Proof

From eqn (6) we may write the vanishing of the total complementary virtual work as

$$\delta\{F\}^{\mathrm{T}}[\phi]\{F\} - \delta\{R\}^{\mathrm{T}}\{d\} = \{0\}. \tag{21}$$

For the case when the redundancy in the system is reduced, we may set those elements of the force vector $\{F\}$ which belong to the removed redundant members, to zero and write for the new system:

$$\delta\{F\}_{\mathrm{r}}^{\mathrm{T}}[\phi]\{F\}_{\mathrm{r}} - \delta\{R\}_{\mathrm{r}}^{\mathrm{T}}\{d\} = \{0\}, \tag{22}$$

where the subscript 'r' refers to the system with reduced degrees of indeterminacy. Now for virtual forces we can take

$$\delta\{F\} = \delta\{F\}_{\mathrm{r}} = \{F\}_{\mathrm{r}} \quad \text{and} \quad \delta\{R\} = \delta\{R\}_{\mathrm{r}}. \tag{23}$$

This can be done since $\delta\{F\}_{\mathrm{r}}$ is certainly an admissible virtual force for the original system. Furthermore, since $\{F\}_{\mathrm{r}}$ also satisfies the equilibrium equations, it may be used as a possible virtual force vector. Now making use of eqn (23) in eqns (21) and (22) and substracting the latter from the former, we find:

$$\{F\}_r^T[\phi](\{F\}-\{F\}_r)=0. \tag{24}$$

Next we expand this expression and noting that $[\phi]$ is symmetric, we write it as

$$\tfrac{1}{2}\{F\}^T[\phi]\{F\}-\tfrac{1}{2}\{F\}_r^T[\phi]\{F\}_r=\tfrac{1}{2}(\{F\}^T-\{F\}_r^T)[\phi](\{F\}-\{F\}_r). \tag{25}$$

Evidently the right-hand side of eqn (25) is positive and the left-hand side is $V^*-V^*_{new}$. This proves the theorem. ∎

Discussion

It is important to note that the above theorem holds when the structure is loaded by prescribed displacements and not forces. Thus, as a redundant member is removed and the structure settles to a new equilibrium configuration, no external work is done on the structure. Under such conditions, the complementary strain energy of the system drops as its degrees of statical indeterminacy is reduced. Finally when the structure becomes a mechanism, its complementary strain energy, under prescribed displacements, vanishes. Proceeding in the opposite direction and increasing the degrees of statical indeterminacy in the system, the complementary strain energy increases and, in the limit as the degrees of indeterminacy of a system approach infinity, that is in the case of a continuum, the complementary strain energy attains its maximum value.

When a continuum is analysed by the force method, usually in terms of stress functions which by definition generate equilibrating force states, the number of terms used in the assumed stress functions correspond to the number of degrees of indeterminacy allowed in the model of the continuum. Thus the larger the number of terms in the assumed stress functions, the larger will be the computed complementary strain energy of the model for the continuum, i.e. the complementary strain energy of the model will be bounded above by the true complementary strain energy of the continuum [see de Veubeke (1964)].

The above theorem holds for linear structural systems. Since for such systems the complementary strain energy is equal in magnitude to the strain energy of the system, the conclusions reached for the complementary strain energy also hold for the strain energy of the system.

For nonlinear systems, two difficulties normally limit the application of the force method of analysis. First the kinematics of deformation appear in the equilibrium equations, preventing satisfaction of the equilibrium equations in terms of force variables alone. This is the consequence of geometric nonlinearities. Second, the constitutive equations relating the member forces and deformations may be not invertible, preventing a definition for V^*. This effect is therefore due to material nonlinearities.

THEOREM 2

This theorem states that when the degrees of statical indeterminacy of a structure are reduced by removing some redundant members, not only is the complementary strain energy of the system reduced (theorem 1) but also this reduction is larger (or at best equal) to the complementary strain energy of the removed members, i.e. there is a net loss of complementary strain energy.

This theorem implies that an indeterminate structure subjected to some prescribed displacements will have a certain amount of complementary energy; this energy is the sum total of the complementary strain energies of the individual members in the structure. Then through removal of a redundant element, the amount of complementary strain energy stored in that element will be removed. At the same time the remaining structure will settle to a new configuration and will lose part of its complementary strain energy (theorem 1). The question that needs to be answered is whether the loss in the structure's complementary strain energy is larger or equal to the complementary strain energy of the redundant element removed. The theorem asserts that:

$$(\Delta V^*)_{\text{structure}} \geqslant (V^*)_{\text{removed members}}. \tag{26}$$

Example

Consider again the system shown in Fig. 1. As before we will remove member 1 and reduce the degrees of indeterminacy of the system from two to one. This will bring about a reduction in the complementary strain energy of the system, as noted earlier, given by

$$(\Delta V^*)_{\text{structure}} = \frac{1}{2}\frac{k_3^2 k_1 d^2}{(k_1+k_2+k_3)(k_3+k_2)}. \tag{27}$$

Now before removing member 1, we found its force as given in eqn (10). Hence the complementary strain energy in this element is given by:

$$V_1^* = \frac{F_1^2}{2k_1} = \frac{k_3^2 k_1 d_1^2}{2(k_1+k_2+k_3)^2}. \tag{28}$$

It now remains to demonstrate that:

$$(\Delta V^*)_{\text{structure}} \geqslant V_1^*.$$

From eqns (27) and (28) we deduce that

$$\frac{1}{(k_1+k_2+k_3)(k_2+k_3)} > \frac{1}{(k_1+k_2+k_3)^2}. \tag{29}$$

The truth of this statement is self evident since $k_1 > 0$.

Proof

Consider a structure under equilibrium and subjected to prescribed displacements $\{d\}$. Now let us remove some redundant members. This will cause the system to undergo some additional deformations but the prescribed displacements will not do further (complementary) work. Hence in this case we may express the principle of complementary virtual work as

$$\delta\{F\}_r^{\mathrm{T}}(\{e\}_r - \{e\}) = 0, \tag{30}$$

where $\{e\}_r$ is the extension vector of the members in the reduced system in the settled new configuration. The vector $\{e\}$ denotes the extensions of the *same* members in the original configuration, i.e. the configuration before the redundant members were removed. In each case, we place zeros for the removed members in correspondence to the zeros in $\{F\}$

Now, we can use the actual forces in the reduced system as admissible virtual forces since the actual forces are also self equilibrating. Thus we may express eqn (30) as

$$\{F\}_r^{\mathrm{T}}\{e\}_r = \{F\}_r^{\mathrm{T}}\{e\}. \tag{31}$$

By invoking the symmetry of the compliance matrix we may express eqn (31) as

$$\{F\}_r^{\mathrm{T}}\{e\}_r = \tfrac{1}{2}\{F\}_r^{\mathrm{T}}\{e\} + \tfrac{1}{2}\{F\}^{\mathrm{T}}\{e\}_r. \tag{32}$$

Now let us add and substract $\frac{1}{2}\{F\}_r^{\mathrm{T}}\{e\}_r$ and $\frac{1}{2}\{F\}^{\mathrm{T}}\{e\}$ to the right-hand side of eqn (32) and rearrange it in the following form:

$$\frac{1}{2}\{F\}_r^T\{e\}_r - \frac{1}{2}\{F\}^T\{e\} = -\frac{1}{2}(\{F\}^T - \{F\}_r^T)(\{e\} - \{e\}_r). \tag{33}$$

We can now recognize the first term on the left-hand side as the complementary strain energy of the reduced system in the new settled equilibrium configuration. The next term is the complementary strain energy of the system before the redundant members were removed, but it excludes the complementary strain energy of the removed members, i.e. this term is the complementary strain energy of the original system less the complementary strain energy of the redundant members. The right-hand side being negative implies that the reduction of the complementary strain energy of the system due to a change in its configuration is larger than the complementary strain energy of the removed redundant members. Hence there is a net loss of complementary strain energy as some redundant members are removed.

CONCLUSIONS AND FURTHER COMMENTS

Two theorems have been introduced and illustrated by simple examples. The conclusions reached are consistent with the basic concepts in linear fracture mechanics for brittle materials, i.e. as a crack grows in a domain subject to prescribed displacements, the magnitude of the complementary strain energy will diminish (theorem 1) and part of the complementary strain energy will be irrecoverable (theorem 2). This irrecoverable energy is used in the generation of a new surface in the crack. Since little can be said, in general, about the complementary strain energy of nonlinear structural systems, it is natural to enquire if the above theorems, which as just noted hold for the *strain energy* of the system in the linear case, remain valid for nonlinear systems. The answer is in the affirmative and below we provide a heuristic demonstration.

The strain energy of a system is generally expressed in terms of continuous displacements, i.e. the notions of statical indeterminacy, redundancy of members and imagined cuts to develop statically determinate systems do not arise explicitly in the displacement formulation. Nevertheless, one can still consider the addition and removal of structural members and note the resulting changes in the strain energy of the system. To this end, consider a statically determinate system as noted conceptually in Fig. 2(a). Evidently, the system will store a certain amount of strain energy when it is subjected to the prescribed displacement d. If we now consider the same system plus a redundant member R (Fig. 2(b)), then we can see that under the same prescribed displacement, the strain energy in the system will be larger, provided the stiffness of R is larger than zero. This result is valid whether the force displacement relationship for R is of the softening or hardening type, but the force displacement curve must possess a positive slope. It is of course possible that under the prescribed displacement the redundant member may remain undeformed. In that case, the strain energy of the system will remain unaltered. Thus in general

$$V_2 \geqslant V_1$$

or

$$V_2 = V_1 + V_{12}.$$

Clearly, addition of further members to the system will bring about a cumulative effect. Thus it is not essential to start from a statically determinate system. In loose terms, then, one can say that the addition of new material (members) will tend to stiffen the system and will cause the system to store a greater amount of strain energy under some prescribed

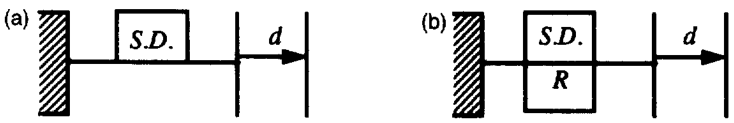

Fig. 2.

displacements. This is the generalization of the first theorem for the strain energy function of nonlinear systems.

To examine the second theorem, it is essential to distinguish the configurations of the system under various states. Let us subject the original system to some prescribed displacements and denote its strain energy by V_1. Next we add the redundant members and subject the structure to the same displacements and obtain the strain energy V_2. Now in this last configuration, i.e. with the redundant members added, we compute the portion of the strain energy in the original system, namely the system before the redundant members were added, and denote it by $V_{(1)}$. For the same configuration we also compute the strain energy in the redundant members and denote it by $V_{(2)}$.

Thus

$$V_2 = V_{(1)} + V_{(2)}.$$

But we also have

$$V_2 = V_1 + V_{12},$$

where V_{12} is the difference in the energy of the two systems and we have, from theorem 1, that

$$V_{12} \geqslant 0.$$

Thus

$$V_{(1)} + V_{(2)} = V_1 + V_{12}$$

and

$$V_{(1)} - V_1 = V_{12} - V_{(2)}.$$

Now we know that $V_{(1)} \geqslant V_1$ since in configuration (1) the original system is not in its lowest strain energy state. The lowest strain energy state is in configuration 1. We thus deduce that

$$V_{12} \geqslant V_{(2)},$$

i.e. the strain energy in the redundant member is less than the energy *difference* between the two systems, i.e. part of the strain energy is lost.

The two theorems we have examined for structural systems have their counterparts in inertial systems where one is concerned with the complementary kinetics rather than the complementary strain energy functions. The corresponding theorem to theorem 1 is known as Bertrand's theorem and the theorem corresponding to theorem 2 is known as Carnot's theorem [see Tabarrok and Rimrott (1994)].

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